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THE TRANSIENT TEMPERATURE DISTRIBUTION IN A SLAB SUBJECTED TO RADIATIVE AND CONVECTIVE HEATING CALCULATED BY VARIATIONAL METHOD

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Abstract.—The variational approach to the heat conduction phenomenon is considered. The use of the variational principle formulated for the system of equations describing the phenomenon, i.e. for Fourier's law and the law of energy conservation is discussed.

The description of the phenomenon is completeted by a balance equation for boundary conditions discussed in the generalized form. This form also makes possible the consideration of nonlinear boundary conditions.

The transient, one-dimensional temperature distribution is determined for plates with radiative and convective heat transfer on the boundary.

NOMENCLATURE

A, region of the body considered;

B, boundary (surface) of the body A;

$Bi_i (i = 0, 1, \ldots, 4)$, generalized Biot numbers defined by equation (38);

B_φ, subsurface of the boundary B;

$c = c(x)$, capacity per unit volume of the body A;

D, D_e, dissipation functions defined by equations (13) and (22), respectively;

F, dimensionless number defined by equation (52);

g_φ, weighting function in balance boundary condition (29);

G, heat flux vector with components $G_i (i = 1, 2, 3)$;

H, heat flow vector with components $H_i (i = 1, 2, 3)$;

I_1, I_2, integrals defined by equations (40) and (41), respectively;

$k = k(x)$, conductivity of the body A;

K, dimensionless number defined by equation (52);

L, dimensionless number defined by equations (53) and (54);

n, normal unit vector with components $n_i (i = 1, 2, 3)$ of surface B taken as positive outwardly;

p_μ, q_ν, generalized coordinates defined in equations (17) and (18), respectively;

$q_0, q_{1\nu}$, generalized coordinates defined in equation (33);

q_1, q_2, q_3, generalized coordinates defined in equation (38);

q_t, generalized coordinate equal to the temperature of the face of the slab when $q_0 = 1$;

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Q_μ, Q_ν, generalized forces defined by equations (25) and (28), respectively;
$\varepsilon_i (i = 0, 1, .., 4)$, constants defined in equations (31) and (32);
η, x/R, dimensionless coordinate;
θ, temperature considered as an increment of the temperature of the body over the absolute temperature T_0 which corresponds to an equilibrium state;
μ_n, root of the characteristic equation (64);
τ, kt/cR^2, dimensionless time;
τ_t, instant of time when $q_0 = 1$;
ψ, potential of the heat flux field defined by equation (14) or equation (16);
ω, constant in equation (39);
ω_1, dimensionless constant defined by equation (61).

Subscripts

f, number of subsurfaces B_φ of the boundary B;
i, describes the direction of the vector coordinate in a rectangular cartesian coordinate system, and has the range of the integers 1, 2, 3;
m, number of the generalized coordinates p_μ;
n, number of the generalized coordinates q_ν and also index of the power in equation (38);
μ, ν, are used to distinguish the generalized coordinates p and q, respectively;
φ, subscript of the subsurface B.

For subscripts the summation convention will be used.

R, semi-thickness of the slab;
t, time;
T, arbitrary reference temperature;
T_0, initial temperature of the slab;
T_a, temperature of the ambient at the face of the slab;
T_g, temperature of the body to which thermal radiation is exposed the face of the considered slab;
T_p, $(u - u_0)/(u_a - u_0)$, dimensionless initial temperature in equation (55);
u, (θ/T), dimensionless temperature;
U, dimensionless quantity defined by equation (60);
u_0, T_0/T, dimensionless initial temperature;
u_a, u_g, T_a/T, T_g/T, dimensionless temperatures of the environment;
V, V_e, thermal potential functions defined by equations (10) and (20), respectively;
$w = w(x, t)$, prescribed heat source in the body A;
x, coordinate vector with components $x_i (i = 1, 2, 3)$;
z, $(q_1 - u_0)/(u_a - u_0)$, dimensionless temperature in equation (55);
z_t, $(q_t - u_0)/(u_a - u_0)$;
Z, Z_a, quantities defined by equations (63) and (59), respectively;
α, β, constants in equation (39);
$\Gamma = \Gamma(\theta, G_i n_i, t)$, function prescribed on the boundary B;
Γ_1, Γ_2, functions given by equations (31) and (32), respectively;
$\delta D^*, \delta V_e^*$, variational invariants defined by equations (11) and (8), respectively.

1. INTRODUCTION

THE ATTENTION of several investigators has been focused on the variational approach to the heat conduction problem, [1–4, 6 and others]. The calculational methods based on this approach are promising and permit us to obtain approximate solutions of the problems which have not yet been solved. These methods permit us, at the same time, to simplify the numerical calculations and allow us to save time when they are adapted to the construction of the numerical calculations on the computer. But, on the other hand, the foundations of the variational principles for heat conduction are not yet clear [4]. Thus appeared in the literature the "restricted-" or "quasi-" variational principles for Fourier's law or for the law of energy conservation, alternatively. Then the heat conduction phenomenon was described by a variational principle and a differential equation considered as a constraint.

In this paper is presented the variational description of the heat conduction problem based on an idea of a system of restricted variational principles [7] formulated for the system of differential equations describing the phenomenon (governing equations), i.e. for Fourier's law and the energy conservation law. Such an approach conducted here for two possible variants of the systems of governing equations gives a basis for putting in order some of the particular forms of the variational principle for the heat conduction problem. There is also shown here how systems of variational principles can be reduced to the forms described in the literature.

The first variant of the variational principle formulated here for time interval dt in which the heat flux vector is used, and the second one for time interval $(0, t)$ in which advantage is taken of the idea of Biot's heat flow vector field [1]. From this formulation as the particular forms follow principles given in [1] and [3].

To complete the description of the phenomenon, the balance equation for boundary conditions is introduced. The form used here for this equation enables us also to take into account the nonlinear boundary conditions.

The use of both variants of variational principles is illustrated by an example of the heating of a slab with radiative and convective heat transfer on the boundary. The results obtained in the form of formulae and graphs are an approximate solution of the problem considered. The accurateness of the method is verified for the particular limiting cases and is compared with others which exist in the literature, with approximate results.

2. PROBLEM FORMULATION

Let us consider the problem of transient heat conduction in an isotropic body A with thermal parameters: $k(x)$—conductivity, and $c(x)$—capacity per unit volume. In the body A is prescribed heat source $w = w(x, t)$ where x denotes coordinate vector with components x_i $(i = 1, 2, 3)$, and t—time. On the boundary B of the body A is prescribed the function

$$\Gamma(\theta, G_i n_i, t) = 0 \text{ on surface } B \tag{1}$$

where θ denotes temperature, G—heat flux vector with components G_i $(i = 1, 2, 3)$, n—normal unit vector with components n_i $(i = 1, 2, 3)$ of surface B taken as positive outwardly.

Let the temperature θ be considered as an increment of the temperature of the body over the absolute temperature T_0 which corresponds to an equilibrium state. At the beginning moment $(t = 0)$ in the region A exists the prescribed temperature field θ_0.

3. VARIATIONAL PRINCIPLE FOR HEAT CONDUCTION

The heat conduction phenomenon in the body A is described by: the law of conservation of energy, and Fourier's law. The first one may be written: (a) for the time interval dt or (b) for the time interval $(0, t)$. Then we obtain alternatively the following systems of equations describing the phenomenon

$$c\dot{\theta} + G_{i,i} = w \tag{2}$$

$$k\theta_{,i} + G_i = 0 \tag{3}$$

for the case (a), and

$$c\theta + H_{i,i} = \int_0^t w\,dt \qquad (4)$$

$$k\theta_{,i} + \dot{H}_i = 0 \qquad (5)$$

for the case (b) where H denotes the heat flow vector field with components $H_i\,(i = 1,2,3)$, introduced by Biot [1, 2].

The foregoing two different systems of partial differential equations are distinguishable to show particular forms of variational principles which are used in the literature by various authors.

The variational principle for the system (2) and (3) takes the form

$$\delta V_e^* + \delta D_e = \int_A w\delta\theta\,dA - \int_B G_i n_i \delta\theta\,dB \quad (6)$$

where the following variational invariants are

$$D_e = \frac{1}{2}\int_A \left(-\frac{1}{k}G_iG_i - 2G_i\theta_{,i}\right)dA \qquad (7)$$

$$\delta V_e^* = \int_A c\dot{\theta}\delta\theta\,dA. \qquad (8)$$

On the other hand, the variational principle for the system (4) and (5) has the form

$$\delta V + \delta D^* = -\int_B \theta\delta H_i n_i\,dB \qquad (9)$$

where

$$V = -\int_A [\tfrac{1}{2}c\theta^2 + \theta(H_{i,i} - \int_0^t w\,dt)]\,dA \quad (10)$$

$$\delta D^* = \int_A \frac{1}{k}\dot{H}_i\delta H_i\,dA. \qquad (11)$$

The quantity V given by (10) may be considered as the canonical form of the thermal potential of the body A. The form

$$V = \tfrac{1}{2}\int_A c\theta^2\,dA \qquad (12)$$

given by Biot [1, 2] can be obtained if the energy balance equation will be treated as the constraint defining the relation between the temperature θ and the vector H_i. The variational principle (6) reduces itself then to the form given in [1, 2] and is equivalent to Fourier's law (5).

By analogy, when we use the equation (3) as the constraint defining the relation between θ and G_i, then the invariant D_e is reduced to the form

$$D_e = \tfrac{1}{2}\int_A k\theta_{,i}\theta_{,i}\,dA \qquad (13)$$

and the variational principle (6) is in this case equivalent to the energy balance equation (2) [3].

The variational principle (6) is equivalent to the system of equations (2) and (3), and it will be equivalent to Fourier's equation for heat conduction if the potential of G_i is introduced [7]

$$G_i = -k\psi_{,i} \qquad (14)$$

with the boundary condition

$$\psi = 0 \text{ on surface } B. \qquad (15)$$

For the variational principle (9), we introduce [1. 2. 6]

$$H_i = -k\psi_{,i} \qquad (16)$$

with boundary condition (15).

It is interesting to notice that with the existence of two variants of the description of the heat conduction phenomenon, we obtain a series of consequences of which we can take advantage in practical calculations. Namely, using one of the reduced forms of the variational principle (6) or (9), we approximate either the energy conservation law or Fourier's law of heat conduction. Thus, we obtain results which approximate, respectively better temperature field or heat flux field in the body considered.

4. LAGRANGIAN FORMULATION

Let the temperature θ be described in terms of n independent parameters $q_\nu = q_\nu(t)$, and G_i (or H_i) in terms of m independent parameters $p_\mu = p_\mu(t)$

$$\theta(x,t) = \theta(x, q_1, q_2, \ldots, q_n) \qquad (17)$$

$$\left.\begin{aligned} G_i(x,t) &= G_i(x, p_1, p_2, \ldots, p_m) \\ H_i(x,t) &= H_i(x, p_1, p_2, \ldots, p_m). \end{aligned}\right\} \quad (18)$$

The system of parameters (generalized coordinates [1]) q_ν and p_μ ($\nu = 1, 2, \ldots, n$; $\mu = 1, 2, \ldots, m$) represents the departure from a certain reference state taken as an origin and for which q_ν and p_μ are equal to zero.

Now, the variational invariants may be expressed as follows [1, 3, 7]

$$\delta V_e^* = \int_A c\dot{\theta}\frac{\partial \theta}{\partial q_\nu}\delta q_\nu \,\mathrm{d}A = \frac{\partial V_e}{\partial \dot{q}_\nu}\delta q_\nu \quad (19)$$

where

$$V_e = \tfrac{1}{2}\int_A c\dot{\theta}^2 \mathrm{d}A \quad (20)$$

$$\delta D^* = \int_A \frac{1}{k}\dot{H}_i\frac{\partial H_i}{\partial p_\mu}\delta p_\mu \mathrm{d}A = \frac{\partial D}{\partial \dot{p}_\mu}\delta p_\mu \quad (21)$$

where

$$D = \frac{1}{2}\int_A \frac{1}{k}\dot{H}_i\dot{H}_i\,\mathrm{d}A. \quad (22)$$

Then, for $n + m$ independent variations δq_ν and δp_μ, the variational principle (6) may be written in the following Lagrangian form

$$\left.\begin{aligned} \frac{\partial V_e}{\partial \dot{q}_\nu} + \frac{\partial D_e}{\partial q_\nu} &= Q_\mu^{(e)} \qquad \mu = 1, 2, \ldots, m \quad (23) \\ \frac{\partial D_e}{\partial p_\mu} &= 0 \qquad \nu = 1, 2, \ldots, n \quad (24) \end{aligned}\right\}$$

where

$$Q^{(e)} = \int_A w\frac{\partial \theta}{\partial q_\nu}\mathrm{d}A - \int_B G_i n_i \frac{\partial \theta}{\partial q_\nu}\mathrm{d}B \quad (25)$$

and the variational principle (9) in the analogical form

$$\left.\begin{aligned} \frac{\partial V}{\partial q_\nu} &= 0 \qquad \mu = 1, 2, \ldots, m \quad (26) \\ \frac{\partial V}{\partial p_\mu} + \frac{\partial D}{\partial \dot{p}_\mu} &= Q_\nu \qquad \nu = 1, 2, \ldots, n \quad (27) \end{aligned}\right\}$$

where

$$Q_\nu = -\int_B \theta\frac{\partial H_i}{\partial p_\mu}n_i\,\mathrm{d}B. \quad (28)$$

The subsystems (23) and (26) are equivalent to the energy balance equation, and the subsystems (24) and (27) to Fourier's law. The subsystems (23) and (27) are composed of the first order ordinary differential equations, and subsystems (24) and (26) of the algebraic equations.

In the practical applications to describe the heat conduction phenomenon in the body, we can choose either equations (23) and (24) or equations (26) and (27). In both cases, it is possible to solve one equation, either (3) or (4), by the cvadrature in the first stage of the calculations for an assumed trial temperature field θ, and next to use the reduced form of the variational principle (6) or (9) to obtain the solution of the problem. In such a procedure, the subsystems (24) or (26) are satisfied identically, and we look for the time history of the generalized coordinates using Lagrangian-type equations (23) or (27).

To use the full forms of the variational principle, we should also introduce trial functions for G_i or H_i and we should next solve simultaneously full system of equations (23) and (24) or (26) and (27).

Choice of the particular form of the trial function for either θ, G_i or H_i depends on the problem considered. Introducing the trial functions in the form of the complete set of the functions, we obtain as the result the exact solution of the problem, analogically as by a classical method [10]. But, it is also convenient, being guided by the physical sense, to introduce simpler forms of the trial functions in the problem under consideration in which generalized coordinates appear having some physical meaning and which can be directly calculated from the Lagrangian-type equation (23) or (27).

5. BOUNDARY CONDITIONS

Let us approximate boundary conditions (1) by the use of the following equations [6, 7]

$$\int_B \Gamma(x, \theta, G_i n_i, t) g_\varphi(x, t)\,\mathrm{d}B = 0 \quad \varphi = 1, 2, \ldots, f \tag{29}$$

where the surface B is divided into f regular subsurfaces B_φ, and g_φ is a prescribed weighting function on each subsurface B_φ. For the variational principle (9), we take into account in (29) the relation: $G_i = \dot{H}_i$.

Let us introduce into the trial functions (17) and (18) an additional set of f generalized coordinates which will be determined by the use of (29). Now, the condition (29) takes the form of an additional system to (26) and (27) of the ordinary differential equations of the first order if the concept of the vector H_i is used, and it takes the form of an additional system to (23) and (24) of algebraic equations if the concept of vector G_i is used.

The vector G_i or H_i can be eliminated from (29) using Fourier's law and we obtain an additional system of algebraic equations in the form [6, 7, 9]

$$\int_{B_\varphi} \Gamma(x, \theta, t) g_\varphi(x, t)\,\mathrm{d}B = 0 \quad \varphi = 1, 2, \ldots, f. \tag{30}$$

The physical meaning of the above-described procedure consists in adjusting the introduced trial function to satisfy the conditions on an average which are prevailing on the boundary of the body. In one-dimensional cases, the satisfaction is exact.

6. APPLICATION TO THE PROBLEM OF THE HEATING OF A SLAB

Let us consider a slab $(0 \leqslant x \leqslant R)$ with constant parameters k and c, and initial temperature $\theta(x, 0) = T_0$. The surface $x = 0$ of the slab is heated by thermal radiation from a body the temperature of which is T_g and by convection from the ambient, the temperature of which is T_a, and the surface $x = R$ is cooled by convection to the ambient, the temperature of which is equal T_0. The boundary conditions can be expressed according to equation (1) in the following form

$$\Gamma_1(0, \theta, G_i n_i) = [G_i n_i - (\varepsilon_2 \theta^n + \varepsilon_1 \theta - \varepsilon_0)] \quad \text{for } x = 0 \tag{31}$$

$$\Gamma_2(R, \theta, G_i n_i) = [G_i n_i - (\varepsilon_3 \theta - \varepsilon_4)] \text{ for } x = R \tag{32}$$

where $\varepsilon_r (r = 0, 1, \ldots, 4)$, and are constants.

We will distinguish two phases in the phenomenon. In the first one, the temperature has not yet begun to rise at the wall $x = R$, and the second one, when it begins to rise.

Let us introduce for the first phase of the phenomenon the following trial function for the temperature distribution

$$u = \begin{cases} \displaystyle\sum_{\nu=1}^{\infty} \left[(q_{1\nu} - u_0)\left(1 - \frac{\eta}{q_0}\right)^{\nu+1} + u_0 \right], & 0 \leqslant \eta \leqslant q_0 \\ u_0 & \eta > q_0 \end{cases} \tag{33}$$

where $u = \theta/T$; $u_0 = T_0/T$; $T > 0$—arbitrary reference temperature; $\eta = x/R$; $q_{1\nu}$, q_0—generalized coordinates: for the dimensionless temperature of the surface $\eta = 0$, and for the penetration depth, respectively.

Applying formulae (13), (20), (25) and (33) for $\eta = 1$ (let $q_{11} = q_1$), we obtain the system of Lagrangian equations (23) (case (a)) in the form

$$q_0[3q_0\dot{q}_1 + 4(q_1 - u_0)\dot{q}_0] - 20(q_1 - u_0) = 0 \tag{34}$$

$$3q_0[2q_0\dot{q}_1 + (q_1 - u_0)\dot{q}_0] - 20(q_1 - u_0) = 0 \tag{35}$$

and respectively, equations (27) (case (b)) in the form

$$q_0[15q_0\dot{q}_1 + 26(q_1 - u_0)\dot{q}_0] - 147(q_1 - u_0) = 0 \tag{36}$$

$$5q_0[2q_0\dot{q}_1 + 3(q_1 - u_0)\dot{q}_0] - 84(q_1 - u_0) = 0 \quad (37)$$

where $\dot{q} \equiv \partial q/\partial \tau$; $\tau \equiv (k/cR^2)t$—dimensionless time.

The first system of the foregoing equations approximates the energy balance equations, and preserves Fourier's law (3), and the second one preserves the law of the conservation of energy, and approximates Fourier's law (5).

To approximate the boundary condition (31) let us use balance equation (30) which takes the form ($f = 1$)

$$2(q_1 - u_0) + (Bi_2 q_1{}^n + Bi_1 q_1 - Bi_0)q_0 = 0 \quad (38)$$

where $Bi_r = (R\varepsilon_r/k)T^{s-1}$ ($r = 1, 2$)—modified Biot's number ($s = 1$ for $r = 1$, and $s = n$ for $r = 2$); $Bi_0 = Bi_2 u_g{}^n + Bi_1 u_a$; $u_g = T_g/T$; $u_a = T_a/T$.

The chosen trial function in the form (33) includes for $f = 1$ two generalized coordinates. The time history of the coordinates should now be determined from a system of two equations. The system consists of the (38) and one optional equation from (34)–(37). The solution of the system can be presented as follows

$$\alpha I_1 - \beta I_2 = -\omega\tau \quad (39)$$

where

$$I_1 = \int_{u_0}^{q_1} \frac{(x - u_0)^2(nBi_2x^{n-1} + Bi_1)}{(Bi_2x^n + Bi_1x - Bi_0)^3}\,dx \quad (40)$$

$$I_2 = \int_{u_0}^{q_1} \frac{x - u_0}{(Bi_2x^n + Bi_1x - Bi_0)^2}\,dx. \quad (41)$$

α, β, ω—are coefficients depending on the system of equations chosen to determine the time history of the generalized coordinates (see Table 1).

The solution of the integrals I_1 and I_2 for $Bi_1 = 0$, and natural n is given in [6]. For $n = 4$ and $Bi_1 \neq 0$ the fourth power polynomial in the denominator can be simply presented as a product of two quadratic forms, and the integrals I_1 and I_2 can be presented as a sum of elementary integrals [9].

The solution given in the form (39) is valid for the body $0 \leqslant x \leqslant R$ up to the time $\tau = \tau_t$ when $q_0 = 1$. The temperature of the surface $x = 0$ at this time reaches the value $q_1 = q_t$ which is an initial value for the second phase of the phenomenon. But we should notice that the formula (39) is also valid for times $0 \leqslant \tau < \infty$ for which it describes the time history of the surface temperature q_1 for the semi-space $x \geqslant 0$. In this case, the asymptotic temperature q_{1as} of the surface $x = 0$ can be found as a real positive root of the polynomial in the denominator of the integral (41) [9]. Thus time $\tau = \tau_t$ establishes the limit of the applicability of formula (39) only in the case of a finite body ($0 \leqslant x \leqslant R$).

Let us approximate the temperature distribution in the second phase of the slab, $0 \leqslant \eta \leqslant 1$, by the following trial function

$$u = q_1 + q_2\eta + q_3\eta^2 + \ldots \quad (42)$$

Thus, we have three generalized coordinates the sum of which determines a temperature of the back side ($x = R$) of the body. To calculate the time history of q_1, q_2, q_3, three equations are needed.

By analogy to the first phase of the heating we obtain the Lagrangian-type equation in the form:
for the case (a)

$$12\dot{q}_3 + 15\dot{q}_2 + 20\dot{q}_1 - 40q_3 = 0 \quad (43)$$

$$3\dot{q}_3 + 4\dot{q}_2 + 6\dot{q}_1 - 12q_3 = 0 \quad (44)$$

$$2\dot{q}_3 + 3\dot{q}_2 + 6\dot{q}_1 - 12q_3 = 0 \quad (45)$$

and for the case (b)

$$45\dot{q}_3 + 70\dot{q}_2 + 126\dot{q}_1 - 126q_3 = 0 \quad (46)$$

$$5\dot{q}_3 + 8\dot{q}_2 + 15\dot{q}_1 - 30q_3 = 0 \quad (47)$$

$$3\dot{q}_3 + 5\dot{q}_2 + 10\dot{q}_1 - 20q_3 = 0 \quad (48)$$

Table 1. Coefficients of the equation (39)

No.	System of equations chosen to determine generalized coordinates	Coefficients		
		α	β	ω
1	(34), (38)	4	7	5
2	(35), (38)	1	3	5/3
3	(36), (38)	26	41	147/4
4	(37), (38)	3	5	21/5

The balance equation (30) for boundary conditions (31) and (32) takes the form ($f = 2$):

$$q_2 = Bi_2 q_1^n + Bi_1 q_1 - Bi_0 \tag{49}$$

$$-(2q_3 + q_2) = Bi_3(q_1 + q_2 + q_3) - Bi_4 \tag{50}$$

where $Bi_3 = R\varepsilon_3/k$; $Bi_4 = Bi_3 u_0$.

The time history of generalized coordinates can be determined by the system of equations (49) and (50), and one arbitrary equation from (43)–(45) and (46)–(48), alternatively.

To compare the case (a) with (b), let us choose systems: (45), (49), (50) and (48), (49), (50). The solution for q_1 can be presented in the form of an integral

$$\int_{q_t}^{q_1} \frac{Bi_2 n x^{n-1} + K + L}{Bi_2 x^n + Kx - F} \mathrm{d}x = -\omega\tau \tag{51}$$

where

$$K = Bi_1 + \frac{Bi_3}{1 + Bi_3}; \quad F = Bi_0 + \frac{Bi_4}{1 + Bi_3} \tag{52}$$

and for the case (a)

$$L = 3\frac{2 + Bi_3^2}{(1 + Bi_3)(4 + Bi_3)}; \quad \omega = 12\frac{1 + Bi_3}{4 + Bi_3} \tag{53}$$

and for the case (b)

$$L = 5\frac{2 + Bi_3^2}{(1 + Bi_3)(7 + 2Bi_3)}; \quad \omega = 20\frac{1 + Bi_3}{7 + 5Bi_3} \tag{54}$$

A detailed discussion of the solutions of the integral (51) is given in [9].

Results of the numerical calculations for the slab are presented on the graphs which illustrate the influence of the dimensionless parameters Bi_i ($i = 0, 1, 2, 3$) and initial temperature u_0 on

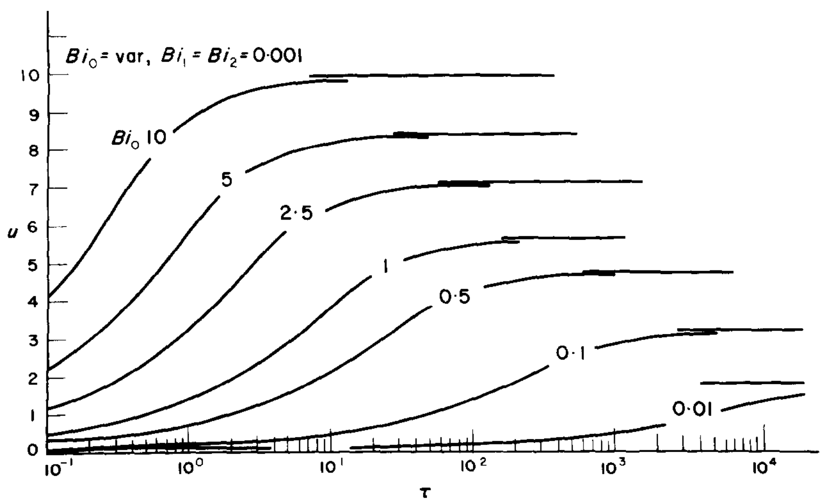

FIG. 1. Transient temperature at the face of a semi-space subjected to radiant and convective heating for various values of the Bi_0 number. The initial temperature $u_0 = 0{\cdot}1$, and coefficients $\alpha = 1, \beta = 3, \omega = 5/3$.

the time history of the temperature q_1. The influence of Bi_0 number which answers for temperatures of the heating medium is illustrated on Fig. 1. Higher asymptotic temperatures q_{1as} correspond to the higher Bi_0 numbers, i.e. higher temperatures u_g and u_a for the same Bi_1, Bi_2 numbers. The influence of Bi_1 and Bi_2 which answer for convection and radiation, respectively is illustrated for a slab and for a semi-space on Fig. 2. The cross section of the temperature graphs for shorter times illustrates the fact that the role of radiation term increases with the increase of the temperature of the body.

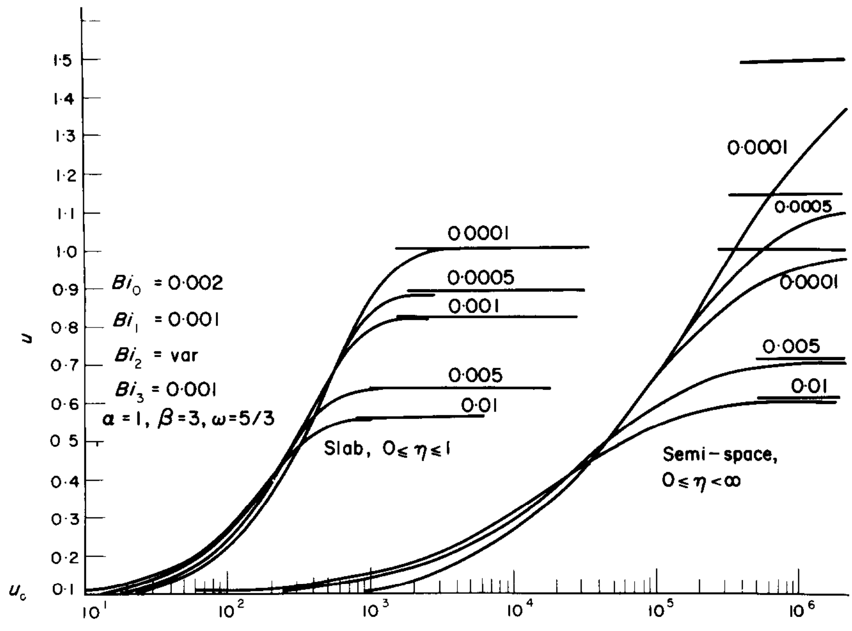

FIG. 2. Transient temperature u at the face of a slab and a semi-space subjected to radiant and convective heating. The back side of the slab is cooled convectively $Bi_0 = 0{\cdot}002$, $Bi_1 = Bi_3 = 0{\cdot}001$, Bi_2 = var; $\alpha = 1$, $\beta = 3$, $\omega = \frac{5}{3}$.

Higher values of the Bi_1, and Bi_2 numbers correspond to a better cooling ability, and then to smaller values of the asymptotic temperature in the body. The role of initial temperature u_0 is presented on Fig. 3. There is also a comparison of the influence of various sets of coefficients α, β, ω on values of the temperature q_1.

The time history of all considered generalized coordinates for $Bi_0 = Bi_1 = Bi_2 = 1$, $Bi_3 = 0{\cdot}5$ and $u_0 = 0$ is illustrated on Fig. 4. One may observe the way in which generalized coordinates approach their asymptotic values.

The values of penetration depth $q_0 > 1$ correspond to the temperature history in the semi-space $x \geqslant 0$.

Division of the heating process on two phases causes a bending of graphs of temperatures in the vicinity of time τ_t.

The results for the transient temperature distribution in a slab subjected to thermal radiation on one face, and insulated at the other one are compared on Fig. 5. with the results obtained by means of a thermal-electrical analog computer [8], and it can be observed that there is a good consistence in the results. It can also be seen that for $N > 20$ the solution for the temperature field in the slab can be limited to the expression derived for the second phase only. (A similar fact can also be observed on Fig. 2.)

For the case of convective heat transfer on

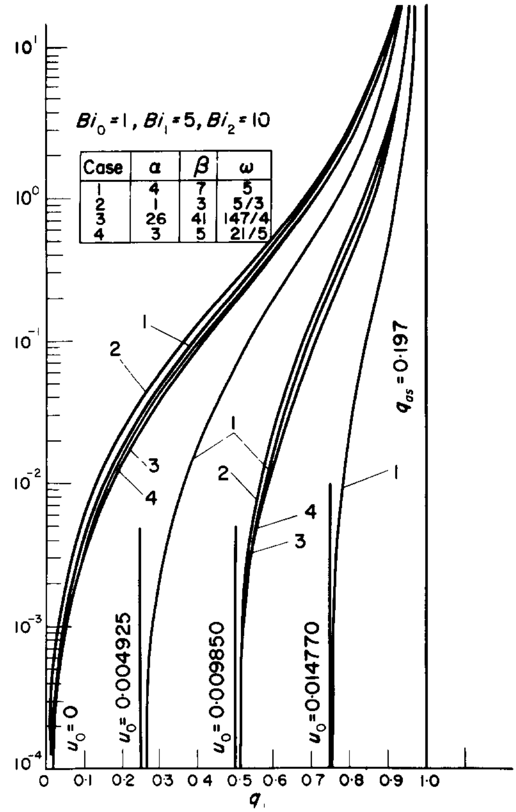

FIG. 3. Transient temperature at the face of a semi-space subjected to radiant and convective heating for various initial temperatures. Cases one to four correspond to the considered variants of formula (39). $Bi_0 = 1$, $Bi_1 = 5$, $Bi_2 = 10$.

both sides of the slab, i.e. for $\varepsilon_2 = 0$ ($Bi_2 = 0$) the solution for the temperature distribution in the slab can be presented as follows:

For the first phase of the heating ($\tau < \tau_t$)

$$T_p = \begin{cases} z\left(1 - \dfrac{\eta}{q_0}\right)^2 & \text{for } 0 \leqslant \eta \leqslant q_0 \\ 0 & \text{for } q_0 < \eta < 1 \end{cases} \tag{55}$$

where

$$T_p = \frac{u - u_0}{u_a - u_0}, \quad z = \frac{q_1 - u_0}{u_a - u_0},$$

$$q_0 = \frac{2z}{Bi_1(1 - z)}.$$

The dependence on time for the normalized dimensionless temperature z is determined from the equation

$$(\beta - \alpha)\ln\frac{1}{1 - z} + (2\beta - 3\alpha)/2 + \frac{2\alpha - \beta}{1 - z} - \frac{\alpha}{2(1 - z)^2} = -\,\omega Bi_1{}^2\tau. \tag{56}$$

The foregoing solution is valid up to time $\tau = \tau_t$ when $q_0 = 1$. This time we derive from the equation (56) putting in it $z = z_t$. The temperature $z_t = (q_t - u_0)/(u_a - u_0)$ of the surface $\eta = 0$ is found by the use of the formula

$$z_t = \frac{Bi_1}{2 + Bi_1}. \tag{57}$$

The coefficients α, β, ω in (56) are taken from Table 1.

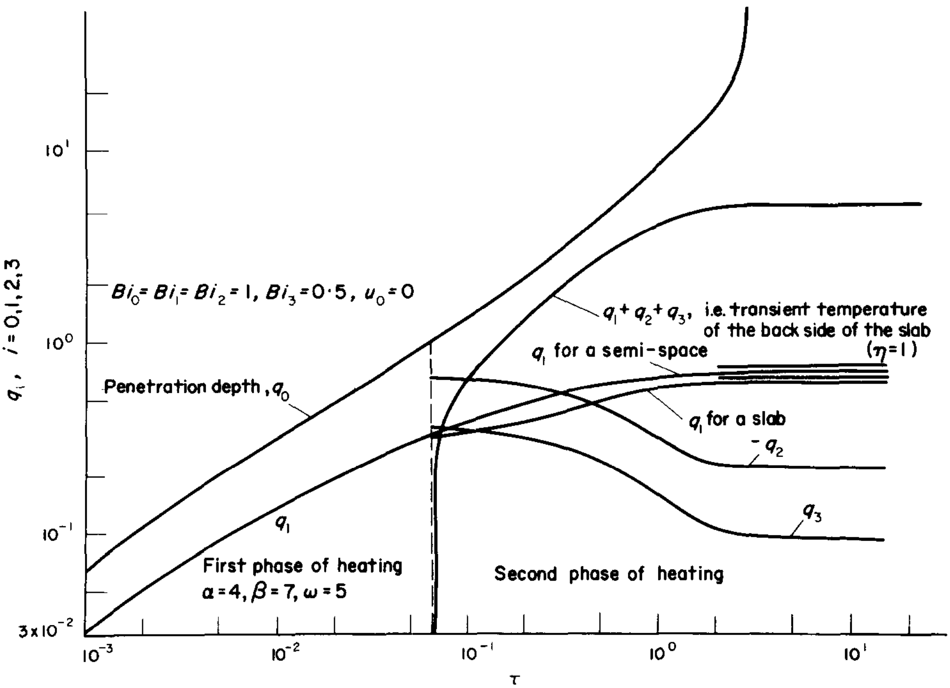

FIG. 4. Time history of generalized coordinates q_i (i = 0, 1, 2, 3) for a slab ($0 \leqslant \eta \leqslant 1$) and a semi-space ($\eta \geqslant 0$) subjected to radiant heating and convective heating at the face. The slab is cooled convectively at the back side. $Bi_0 = Bi_1 = Bi_2 = 1, Bi_3 = 0{\cdot}5, u_0 = 0$.

For the second phase of the heating (for $\tau \geqslant \tau_t$)

$$T_p = z_t + \frac{Bi_1 + Bi_1 Bi_3(1-\eta) - U}{Bi_1 + Bi_3 + Bi_1 Bi_3}(1 - Z_a) \tag{58}$$

where

$$Z_a = \frac{Bi_1(1 + Bi_3) - U}{Bi_1 + Bi_1 Bi_3(1-\eta) - U} \times \left(1 + Bi_1\eta - \frac{Bi_1 + Bi_3 + Bi_1 Bi_3}{2 + Bi_3}\eta^2\right) \times e^{-\omega_1 \tau} \tag{59}$$

$$U = z_t(Bi_1 + Bi_3 + Bi_1 Bi_3) \tag{60}$$

$$\omega_1 = \frac{K\omega}{K + L}. \tag{61}$$

The difference between case (a) and (b) appears in ω_1.

The exact solution for the convective heat transfer is [5]

$$T = \frac{Bi_1 + Bi_1 Bi_3(1-\eta)}{Bi_1 + Bi_3 + Bi_1 Bi_3}(1 - Z) \tag{62}$$

where

$$Z = \sum_{n=1}^{\infty} \frac{\cos[\mu_n(1-\eta)] + Bi_3 \sin[\mu_n(1-\eta)]/\mu_n}{\left(1 + \frac{Bi_3}{Bi_1}\right)\frac{\sin\mu_n \cos\mu_n + \mu_n}{2\sin\mu_n} + \frac{Bi_3 \sin\mu_n}{\mu_n}} e^{-\mu_n^2 \tau} \tag{63}$$

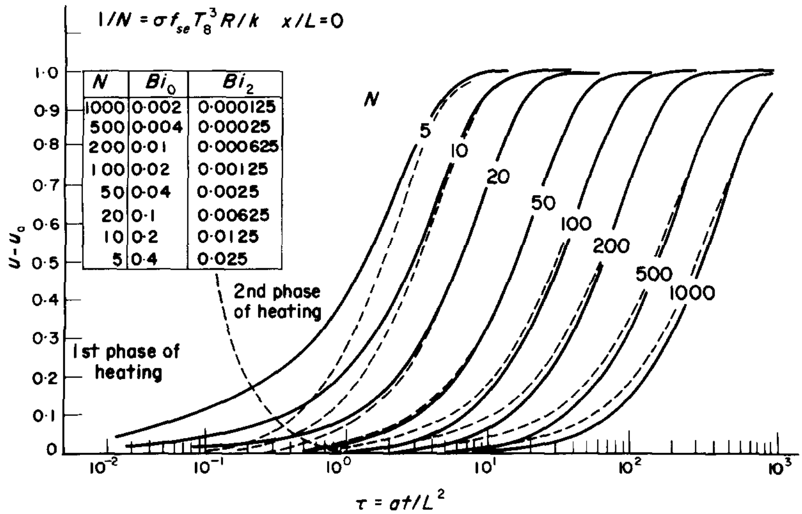

FIG. 5. Transient temperature at the insulated face of a slab subjected to radiant heating when $\theta_0/T_g = 0{\cdot}5$ ($T_a = \theta_0$). ——— Variational solution. ----- Solution obtained by means of a thermal-electric analog computer.

μ_n are the roots of

$$\cot \mu = \frac{\mu^2 - Bi_1 Bi_3}{\mu(Bi_1 + Bi_3)}. \tag{64}$$

The set of sinus and exponential functions (63) appearing in the exact solution (62) is approximated by one exponential function (59) in the approximate solution (58). Results obtained by use of the formulae (55), (58) and (62) for the convective heat transfer are presented on graphs (Figs. 6–9) for various Biot numbers. Bendings which may be seen on them for a certain instant of time are the consequence of the dividing of the phenomenon in two phases. We may also observe the average character of the approximate solutions.

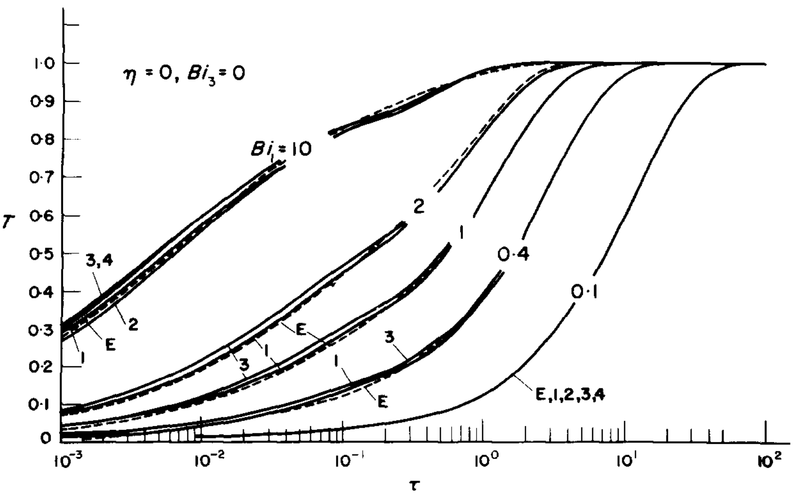

FIG. 6. Temperature response at the front face of a slab suddenly exposed to a uniform temperature convective environment (E—exact solution; 1, 2, 3, 4—approximate solutions —numbers according to Table 1).

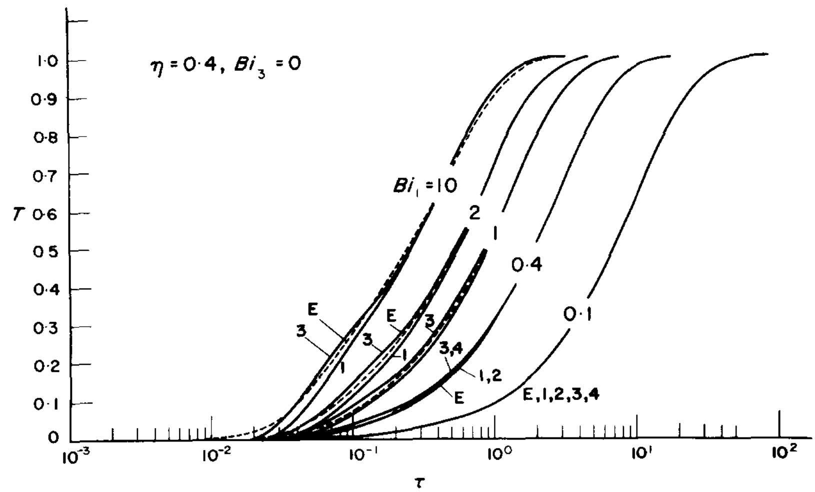

FIG. 7. Temperature response for $\eta = 0{\cdot}4$ of the slab, $0 \leqslant \eta \leqslant 1$, with insulated back face $\eta = 1$ sudden exposed to a uniform-temperature convective environment (E— exact solution; 1, 2, 3, 4—approximate solutions—numbers according to Table 1).

7. CONCLUSIONS

It has been shown that two different ways of constructing the variational principle for heat conduction is possible. Thus, we obtain variational principles (6) and (9) in which both temperature and heat flux vector (G_i) or temperature

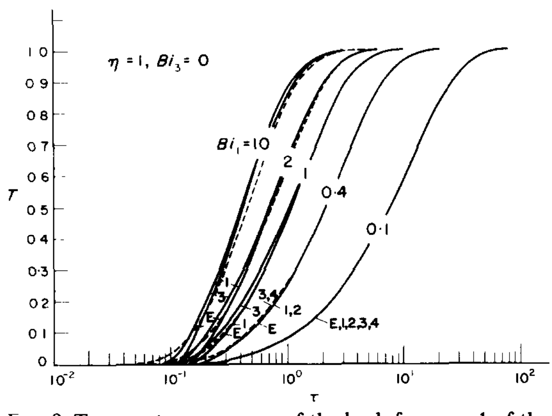

FIG. 8. Temperature response of the back face $\eta = 1$ of the slab, $0 \leqslant \eta \leqslant 1$, sudden exposed to a uniform-temperature convective environment (E—exact solution; 1, 2, 3, 4—approximate solutions—numbers according to Table 1).

and heat flow vector (H_i) appear. These principles may be considered as being in canonical form and can be reduced to the particular forms described in literature [1, 3] when assumptions are made between heat flux (or heat flow) vector and temperature.

The variational principle completed by balance equation (29) for boundary conditions permits us to solve heat conduction problems with nonlinear boundary conditions which were illustrated for the case of one-dimensional bodies.

The obtained results indicate some available ways which may be chosen when the variational approach is preferable. The trial functions for temperature field and for the vector field G_i or H_i can be introduced in the considered body, and the full, canonical form of the variational principle can be used. However, it is convenient to employ one of two particular forms with suitable constraints.

The results obtained by use of Biot's method based on the heat flow vector field, and the results obtained by the method based on the variational principle for the law of energy conservation seem to be similar with acurate approximation of the problem considered. However, the latter one has the advantage of simplicity because it does not need the introduction

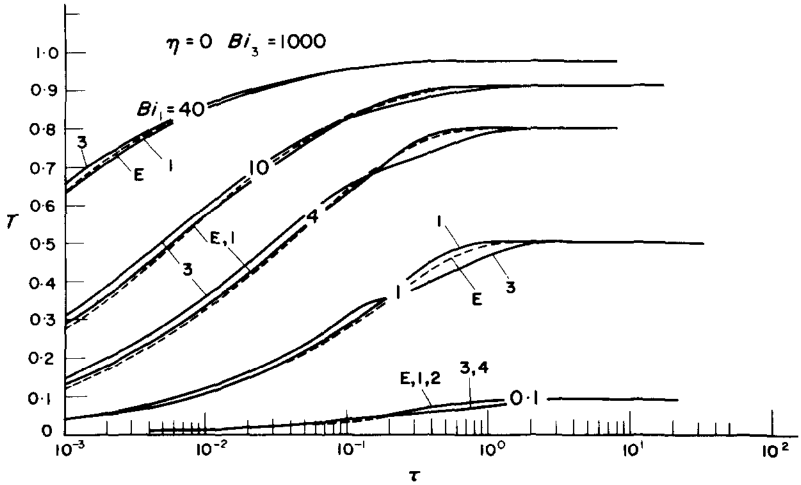

FIG. 9. Temperature response of the front face $\eta = 0$ of the slab, $0 \leqslant \eta \leqslant 1$, with convective heat transfer on both sides after step rise of the temperature of the front-face convective environment (E—exact solution; 1, 2, 3, 4—approximate solutions—numbers according to Table 1).

of the additional potential field. Then in some cases it enables us to solve more complicated problems. Thus, the example of the heating of a slab by radiative and convective heat transfer on the surface could be reconsidered for the case of cylindrical and spherical geometry.

Improving the accuracy of the approximate solutions is possible by a better adjustment of the trial function to the problem considered, e.g. increasing the number of the generalized coordinates.

Variational principles discussed in this paper are based on Biot's idea of quasi-variational principle [1, 2]. This idea is connected with the proper choice of particular forms of dependence on generalized coordinates of the temperature field and heat flux field to satisfy the relations (19) or (21).

It is also possible to have a different approach to the problem and to have the formulation of a convolution variational principle which does not need such assumptions [10]. But the Lagrangian type equations considered in the present paper also follow from the convolution theory.

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APPENDIX A

The solution for the phase (a) for a slab given by formula (39) can be simplified by application of $y = x_1/B$ where $x_1 = x - u_0$ and $B = Bi_2x^n + Bi_1x - Bi_0$. Then we obtain

$$I = \alpha I_1 - \beta I_2 = -y_3^2/2 + (\alpha - \beta)I_2 \tag{A.1}$$

where $y_3 = y|_{x=q_1}$.

APPENDIX B

Integral I_2 can be presented for $n = 4$ by means of elementary integrals, namely we have

$$I_2 = \int_{u_0}^{q_1} C_n \, dx \quad \text{where} \quad C_n = \frac{x - u_0}{(Bi_2x^n + Bi_1x - Bi_0)^2} \tag{B.1}$$

and for $n = 4$ we have

$$C_4 = \frac{x - u_0}{\prod_{i=1}^{2} (x^2 + a_ix - b_i)^2 Bi_2^2} \tag{B.2}$$

where

$a_1 = y^{\frac{1}{2}} = \mu; \quad a_2 = -\mu; \quad b_1 = (K - \mu^3)/2\mu;$

$b_2 = -(K + \mu^3)/2\mu.$

Expression (B.2) can be presented in the form of the following sum:

$$C_4 = \frac{1}{Bi_2^2}\left[\frac{c_1x + c_2}{x^2 + a_1x + b_1} + \frac{c_3x + c_4}{(x^2 + a_1x + b_1)^2} + \frac{c_5x + c_6}{x^2 + a_2x + b_2} + \frac{c_7x + c_8}{(x^2 + a_2x + b_2)^2}\right]. \tag{B.3}$$

Coefficients c_j can be calculated from the system

$$\{a_{ij}\}\{c_j\} = \{b_i\} \quad i, j = 1, 2, \ldots, 8 \tag{B.4}$$

where

$\{b_i\}^T = \{0, 0, 0, 0, 0, 0, 1, -u_0\};$

$a_{i,i} = a_{i,i+4} = 1$ for $i = 1, 2, 3, 4;$

$a_{2,5} = a_{3,6} = -a_{2,1} = -a_{3,2} = \mu;$

$a_{4,7} = a_{5,8} = -a_{4,3} = -a_{5,4} = 2\mu;$

$a_{4,1} = a_{5,2} = a_{5,6} = a_{4,5} = K;$

$a_{5,3} = a_{6,4} = (K + 2\mu^3)/\mu;$

$a_{3,1} = a_{4,2} = -b_2;$

$a_{3,5} = a_{4,6} = -b_1;$

$a_{5,1} = a_{6,2} = -(F + K\mu);$

$a_{5,5} = a_{6,6} = -F + K\mu;$

$a_{7,5} = a_{8,6} = Fb_1;$

$a_{7,1} = a_{8,2} = Fb_2;$

$a_{6,1} = a_{7,2} = (2F\mu^2 + K^2 + K\mu^3)/2\mu;$

$a_{6,5} = a_{7,6} = -(2F\mu^2 + K^2 - K\mu^3)/2\mu;$

$a_{6,3} = a_{7,4} = -(K + \mu^3);$

$a_{6,7} = a_{7,8} = -K + \mu^3;$

$a_{7,3} = a_{8,4} = b_2^2;$

$a_{7,7} = a_{8,8} = b_1^2;$

and the other terms of the matrix $\{a_{ij}\}$ vanish, and:

$$K = Bi_1/Bi_2 \text{ and } F = Bi_0/Bi_2.$$

Integral I_2 by use of expression (B.3) can be presented as follows

$$I_2 = \frac{1}{Bi_2^2}\left[\frac{(2c_4 - c_3a_1)q_1 + c_4a_1 + 2c_3b_1}{-\delta_1(q_1^2 + a_1q_1 - b_1)} - \frac{(2c_4 - c_3a_1)u_0 + c_4a_1 + 2c_3b_1}{-\delta_1(u_0^2 + a_1u_0 - b_1)} + \frac{(2c_8 - c_7a_2)q_1 + c_8a_2 + 2c_7b_2}{-\delta_2(q_1^2 + a_2q_1 - b_2)} - \frac{(2c_8 - c_7a_2)u_0 + c_8a_2 + 2c_7b_2}{-\delta_2(u_0^2 + a_2u_0 - b_2)} + 0{\cdot}5c_1 \ln\left|\frac{q_1^2 + a_1q_1 - b_1}{u_0^2 + a_1u_0 - b_1}\right| + 0{\cdot}5c_5 \ln\left|\frac{q_1^2 + a_2q_1 - b_2}{u_0^2 + a_2u_0 - b_2}\right| + \frac{\delta_1(c_1a_1 - 2c_2) + 4c_4 - 2c_3a_1}{-2\delta_1^{\frac{3}{2}}} \ln\left|\frac{(2q_1 + a_1 - \delta_1^{\frac{1}{2}})}{(2q_1 + a_1 + \delta_1^{\frac{1}{2}})}\frac{(2u_0 + a_1 + \delta_1^{\frac{1}{2}})}{(2u_0 + a_1 - \delta_1^{\frac{1}{2}})}\right| - \frac{\delta_2(2c_6 - c_5a_2) - 2c_8 + c_7a_2}{(-\delta_2)^{\frac{1}{2}}} \times \left(\arctan\frac{2q_1 + a_2}{(-\delta_2)^{\frac{1}{2}}} - \arctan\frac{2u_0 + a_2}{(-\delta_2)^{\frac{1}{2}}}\right)\right]$$

where

$$\delta_1 = a_1^2 + 4b_1; \quad \delta_2 = a_2^2 + 4b_2.$$

DISTRIBUTION DE TEMPERATURE TRANSITOIRE CALCULEE PAR UNE METHODE VARIATIONNELLE DANS UNE PLAQUE SOUMISE A UN CHAUFFAGE PAR RAYONNEMENT ET CONVECTION

Résumé—On considére la description variationelle du phenoméne de conduction de chaleur. On examine une application du principle variationel pour un système des équations décrivant le phénomène, c'est a dire la loi de Fourier et las loi de conservation d'energie.

La description du phénomène est completée par la condition déquilibre considérée dans une forme générale. Il est einsi possible de considérer les conditions aux limites nonlinéaires.

On a determiné la distribution de temperature dans le cas de conduction de chaleur unidimensionelle en régime transitoire dans des parois planes chauffes par convection et rayonnement de la chaleur.

On a determiné la distribution de temperature dans le cas de conduction de chaleur unidimensionelle en régime transitoire dans des parois planes chauffes par convection et rayonnement de la chaleur.

DIE IN STATIONARE TEMPERATURVERTEILUNG IN EINER BESTRAHLTEN UND KONVEKTIV BEHEIZTEN PLATTE, BERECHNET NACH EINER VARIATIONSMETHODE

Zusammenfassung—Es wird die Anwendung des Variationsprinzips auf Probleme der Wärmeleitung betrachtet. Hierbei wird das Variationsprinzip für die das System beschreibenden Gleichungen formuliert, nämlich das Gesetz von Fourier und der Erhaltungssatz der Energie.

Zusätzlich wird eine Bilanzgleichung für die Randbedingungen in allgemeiner Form aufgestellt, mit deren Hilfe man auch nichtlineare Randebedingungen behandeln kann.

Die nichstationäre, eindimensionale Temperaturverteilung in einer Platte wird berechnet mit Wärmeübetragung durch Strahlung und Konvektion an ihren Rändern.

ВАРИАЦИОННЫЙ МЕТОД РАСЧЕТА НЕСТАЦИОНАРНОГО РАСПРЕДЕЛЕНИЯ ТЕМПЕРАТУР В ПЛИТЕ ПОДВЕРГАЕМОЙ ЛУЧИСТОМУ И КОНВЕКТИВНОМУ НАГРЕВУ

Аннотация—В работе принят вариационный подход к описанию процесса теплопроводности. Рассматривается применение вариационного принципа сформулированного для системы уравнений описывающей процесс теплопроводности, то есть для закона Фурье и баланса энергии.

Описание процесса дополнено уравнением баланса дла граничных условий представленных в общей форме. Форма эта дает тоже возможность рассматривать нелинейные граничные условия.

Определено однокоординатное распределение температуры для нестационарного режима в плитах, с учетом радиационного и конвекционного теплообменов на поверхности.